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RANDOM WALK MODELS FOR TARGET TRACKING

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Knowledge of the position of a mobile target has often to be ascertained from knowledge of the position at an earlier time and a model of the target motion. The models considered in this research contribution are based on random walk theory and are applicable where the motion is of a complex, time-varying character. Differences in the initial conditions assumed and minor variations in the model account for the different sections. In each case, density and distribution functions are derived for the coordinates of the target.

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INTRODUCTION

If a set of position-related observations are made simultaneously on a target of interest, the joint position density posterior to the data resulting from the observations provides all of the information available and the basis for any action to be taken. If the observations are not simultaneous, or if some time has elapsed since the observations were made, and, in addition, the target is mobile, the introduction of a model to account for target motion becomes necessary.

For many applications, the motion may be conveniently characterized by the joint density of target course and speed, or, more precisely, by the joint density of target course and speed during the particular time interval of interest. If the characterization that is appropriate for this time interval is independent of time, the problem of specifying the position density at the end of the interval is greatly simplified. A treatment in terms of space variables alone is then possible and is achieved by replacing the given joint density of course and speed by that of course and range traversed in the time interval that is applicable. This is the approach used in reference 1 where the assumption of stationarity is justified by the application.

In many cases, however, the speed or course, or both, will change during the interval of interest. The interval may be very long, the target may be confined to a limited area or engaged in a maneuvering tactic. In such cases, the model for the motion must take these changes into account. If, however, the principal concern is the final position of the target, only the cumulative effect of the changes which take place during the time interval contributes to this. Thus, if the vector sum of all the component motions can be determined, the problem is essentially reduced to the stationary case referred to above.

For the application at present under consideration, certain assumptions may be made. The time intervals and target speeds are such that a flat-earth approximation is adequate. The components of the motion are assumed to be mutually independent and sufficiently large in number during any time interval that will occur to justify the application of the Central Limit Theorem. Changes in course will, for the greater part of the research contribution, be uniformly distributed. The principal problem to be addressed consists of finding the final position density after a time interval of arbitrary length, during which the target motion is of a time-varying nature, given that the initial position is normally distributed. In addition to this problem, the research contribution provides numerous related results. All of these results are based on contributions to the theory of wave propagation by Rayleigh, Rice, Nakagami, and others.

PRELIMINARY REMARKS

The initial position of the target is designated (x, y) and the joint density function $p_{xy}(x, y)$ is assumed known.

The motion of the target during any time interval T is assumed to consist of n segments where n is large. An arbitrary segment, say the jth, is characterized by speed V_j, course φ_j, and duration τ_j or, alternatively, by distance traversed or range $r_j = V_j \tau_j$ and course φ_j. In addition to the independence of different segments, the variables V_j and r_j will be assumed independent of φ_j, and the latter will be assumed to be uniformly distributed on the interval $0 \leq \varphi_j < 2\pi$, for all j, throughout most of the research contribution. Only such limitations as are required for the application of the Central Limit Theorem are imposed on the densities $p_{r_j}(r_j)$ and $p_{V_j}(V_j)$. The projections of the component traverses in the directions of the reference coordinate axes are given by:

$$x_j = r_j \cos \varphi_j = V_j \tau_j \cos \varphi_j$$

$$y_j = r_j \sin \varphi_j = V_j \tau_j \sin \varphi_j \quad .$$

The cartesian and polar coordinates at the end of the time interval T are denoted by (u, v) and (R, θ) respectively, and it is the density functions of these variables for which formal expressions are required. The relationship of these variables to those previously defined is illustrated in figure 1.

Note that (R, θ) does not describe the closing vector of the random motion unless the initial position coincides with the origin of the coordinate system. It is clear, however, that if an additional vector, equivalent in effect to the initial conditions, is included as part of the motion, the final position density will be unchanged, but (R, θ) will then describe the closing vector of the augmented but less homogeneous model of the motion so obtained.

The approach that will be taken is to consider a series of problems of increasing complexity and to derive expressions relating to the final position probabilities in each case. These expressions will contain parameters whose values may be ascertained from those used to characterize the random motion. For the application of interest, the total duration of the motion T will be known, and for this case we shall assume that the mean-square speed V_T^2 is also specified. This is defined as:

$$V_T^2 = \frac{1}{T} \int_0^T E\{V^2(t)\}\, dt \quad .$$

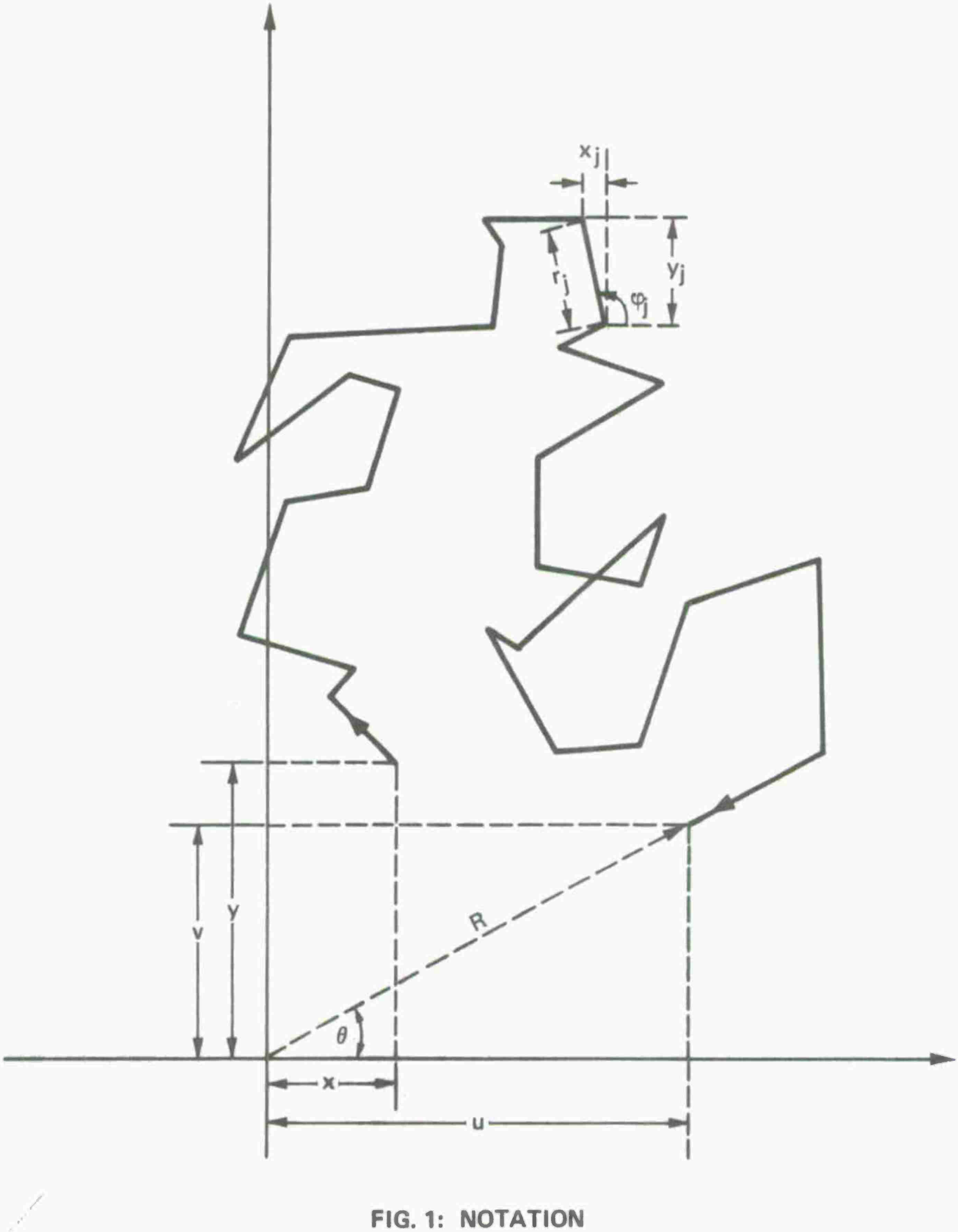

FIG. 1: NOTATION

Since, however,

$$V(t) = V_j \qquad \text{for } t_{j-1} < t \le t_j$$

$$\tau_j = t_j - t_{j-1}$$

$$T = \sum_{j=1}^{n} \tau_j \quad ,$$

this may also be written as:

$$V_T^2 = E\left[\frac{1}{T} \sum_{j=1}^{n} V_j^2 \tau_j\right]$$

where E denotes expectation.

We shall also consider the case in which T is not specified, but the total number of segments of the motion, N , is known. While this case has no immediate bearing on the application of current interest, the results are similar in form involving only a redefinition of key parameters. For this case, in addition to N , we shall assume that the mean square range:

$$r_N^2 = \frac{1}{N} \sum_{j=1}^{N} E(r_j^2)$$

is specified.

PROBLEM 1

Problem 1 is a special case of the problem of primary interest. The initial position is known and coincides, or may be made to coincide, with the origin of the coordinate system. The variables $r_j = V_j \tau_j$ are identically distributed for all j , and

$$p_{\varphi_j}(\varphi_j) = \frac{1}{2\pi} \qquad 0 \le \varphi_j < 2\pi \quad .$$

The total duration of the motion T and the mean square speed V_T^2 are specified.

$$u = \sum_{j=1}^{n} V_j \tau_j \cos \varphi_j = \sum_{j=1}^{n} x_j$$

$$v = \sum_{j=1}^{n} V_j \tau_j \sin \varphi_j = \sum_{j=1}^{n} y_j \quad .$$

It follows from these relationships that the variables x_j and y_j are also identically distributed, and, since n is large, the conditions of the Central Limit Theorem are satisfied. The coordinates of the final position u and v are therefore approximately normal, and the parameters of the density are obtained as follows.

For any time t,

$$\varphi(t) = \int_0^t \varphi(t')dt' = \sum_{j=1}^{n'} \varphi_j + \varphi_0$$

where n' is the integral number of segments traversed in the time t. But since the sum, modulo 2π, of any number of random variables uniformly distributed on the interval $0 \le \varphi_j < 2\pi$ is uniformly distributed on the same interval, we have

$$p_{\varphi(t)}\{\varphi(t)\} = \frac{1}{2\pi} \qquad 0 \le \varphi(t) < 2\pi \quad .$$

Also, since $V(t)$ and $\varphi(t)$ are independent, and

$$u = \int_0^T V(t) \cos \varphi(t)\, dt$$

$$v = \int_0^T V(t) \sin \varphi(t)\, dt$$

$$E(u) = \int_0^T E\{V(t)\} E\{\cos \varphi(t)\}\, dt$$

$$= 0$$

since

$$E\{\cos \varphi(t)\} = E\{\cos \varphi_j\} = \frac{1}{2\pi} \int_0^{2\pi} \cos \varphi_j d\varphi_j = 0 \quad .$$

Similarly,

$$E(v) = 0 \quad .$$

$$\sigma_u^2 = E(u^2) = \int_0^T \int_0^T E\{V(t)\ V\ (t')\}\ E\{\cos \varphi(t) \cos \varphi(t')\}\, dt\, dt'$$

$$= T \int_0^T E\{V^2(t)\}\ E\{\cos^2 \varphi(t)\}\, dt$$

$$= \frac{T}{2} \int_0^T E\{V^2(t)\}\, dt$$

since

$$E\{\cos^2 \varphi(t)\} = E\{\cos^2 \varphi_j\} = \frac{1}{2\pi} \int_0^{2\pi} \cos^2 \varphi_j \, d\varphi_j = \frac{1}{2} \quad .$$

Therefore,

$$\sigma_u^2 = E(u^2) = \frac{T^2}{2} V_T^2 \quad ,$$

and, similarly, it may be shown that:

$$\sigma_v^2 = E(v^2) = \frac{T^2}{2} V_T^2$$

and that:

$$E(uv) = 0 \quad .$$

Introducing the additional notation:

$$\sigma^2 = \frac{T^2}{2} V_T^2$$

$$\alpha = 2\sigma^2 \quad ,$$

we obtain

$$p_u(u) = (2\pi)^{-\frac{1}{2}} \sigma^{-1} \exp(-u^2/2\sigma^2)$$

$$p_v(v) = (2\pi)^{-\frac{1}{2}} \sigma^{-1} \exp(-v^2/2\sigma^2)$$

$$p_{uv}(u, v) = p_u(u) p_v(v) \quad ,$$

$$= (2\pi)^{-1} \sigma^{-2} \exp\{-\frac{1}{2\sigma^2}(u^2 + v^2)\}$$

$$p_{R\theta}(R, \theta) = R\, p_{uv}(R\cos\theta,\ R\sin\theta)$$

$$= (2\pi)^{-1} \sigma^{-2} R \exp(-R^2/2\sigma^2)$$

$$= (\pi\alpha)^{-1} R \exp(-R^2/\alpha) \qquad 0 < R < \infty$$

$$0 \le \theta < 2\pi$$

$$p_\theta(\theta) = (\pi\alpha)^{-1} \int_0^\infty R \exp(-R^2/\alpha)dR$$

$$= (2\pi)^{-1} \qquad 0 \le \theta < 2\pi$$

$$p_R(R) = 2(\alpha)^{-1} R \exp(-R^2/\alpha) \qquad 0 < R < \infty \quad .$$

The cartesian coordinates of the final position have therefore a bivariate normal density. The marginal density of θ is uniform, as might have been expected. The distance from the origin to the final position is Rayleigh distributed with mean square value α.

The distribution function for R is:

$$p_R(R) = 2(\alpha)^{-1} \int_0^R u \exp(-u^2/\alpha)\, du$$

$$= 1 - \exp(-R^2/\alpha) \quad .$$

Hence,

$$\text{Prob}\,(R > \rho) = \exp(-\rho^2/\alpha) \quad .$$

But, since $\alpha = E(R^2)$

$$R_{RMS} = \alpha^{\frac{1}{2}}$$

and

$$\text{Prob}\left(\frac{R}{R_{RMS}} > \rho\right) = \exp(-\rho^2) \quad .$$

Contours of constant probability are concentric circles centered at the origin. In fact, all of the expressions obtained exhibit this central symmetry as a consequence of the fact that, in this problem, both the motion and the initial conditions have this property.

PROBLEM 1a

This problem differs from Problem 1 only in the following respect. Instead of the duration of the motion and the mean square speed, the total number of segments, N, and the mean square range r_N^2, are specified.

$$u = \sum_{j=1}^{N} r_j \cos \varphi_j = \sum_{j=1}^{N} x_j$$

$$v = \sum_{j=1}^{N} r_j \sin \varphi_j = \sum_{j=1}^{N} y_j \, .$$

As before, the conditions of the Central Limit Theorem are satisfied. Taking expectations, therefore, we obtain:

$$E(u) = E(v) = 0$$

$$E(u^2) = E(v^2) = \frac{N}{2} E\,(r_j^2) = \frac{N^2}{2} r_N^2$$

$$E(uv) = 0 \quad .$$

If, therefore, we redefine the parameter σ^2 for this problem by:

$$\sigma^2 = \frac{N^2}{2} r_N^2 \quad ,$$

all of the results obtained in Problem 1 hold for this problem also.

Further, although a characterization of the motion in terms of its duration and mean square speed will be assumed for the remainder of the problems we shall consider, it is clear from the above that formally identical solutions will hold for a characterization in terms of total number of segments and mean square range. The inclusion of this problem is to indicate the single modification that is necessary.

PROBLEM 2

The statement of Problem 2 is similar to that of Problem 1 with one exception. The variables r_j are not identically distributed for all j .

In the treatment of the previous problems it was possible to ascertain the densities of final position variables through the application of the Central Limit Theorem without reference to the properties of the density $p_{r_j}(r_j)$. This theorem, however, holds under more general conditions than those imposed, so all of the results obtained remain valid when the variables r_j have different distributions for different values of j , yet satisfy this more general constraint.

While a precise statement of the most general conditions under which the Central Limit Theorem holds is available (reference 2), it is doubtful if it would serve in the present context. Essentially what is required is that the variance of any one term of the process is negligible in comparison to the variance of their sum. Thus, if the variables r_j are not identically distributed, the results obtained in Problem 1 remain valid if:

$$E(r_j^2) = E(V_j^2 \tau_j^2) << T^2 V_T^2$$

for all j , and in Problem 1a if:

$$E(r_j^2) << N\, r_N^2$$

for all j .

These conditions may therefore replace the requirement that the component ranges have identical distributions in Problems 1 and 1a respectively.

To complete the discussion of Problem 2, we consider how to proceed when either of these conditions is not satisfied. If the condition is not met for a single value of j , the contribution of this component may be separated from the others whose combined effect can be ascertained via the Central Limit Theorem. This is clearly equivalent to the addition of a random term to the motion described in Problem 1. Motions with this feature will be considered later. Where the condition is not met for more than a single value of j , the sum of such terms may be treated separately, so this situation is not essentially different.

PROBLEM 3

In Problem 3 the initial position of the target is arbitrary but is known with certainty. All other factors are as in Problem 1 or Problem 2.

For an initial position (x, y)

$$u = x + \int_0^T V(t) \cos \varphi(t)\, dt$$

$$v = y + \int_0^T V(t) \sin \varphi(t)\, dt \quad .$$

It is clear from the preceding problems that:

$$E(u) = x$$

$$E(v) = y$$

$$\sigma_u^2 = \sigma_v^2 = \sigma^2$$

$$\rho_{uv}\sigma_u\sigma_v = E\{(u-x)(v-y)\} = 0 \quad .$$

Therefore,

$$p_{uv}(u, v) = p_u(u)p_v(v)$$

$$p_{R\theta}R, \quad = (2\pi)^{-1}\sigma^{-2} \exp\{-(u-x)^2/2\sigma^2 - (v-y)^2/2\sigma^2\}$$

$$p_{R\theta}(R, \theta) = Rp_{uv}(R \cos \theta,\ R \sin \theta)$$

$$= (\pi\alpha)^{-1} R \exp\left[-\frac{1}{\alpha}\{R^2+r^2-2Rr \cos(\theta-\varphi)\}\right]$$

where

$$r = (x^2+y^2)^{\frac{1}{2}}$$

$$\varphi = \arctan y/x \quad .$$

$$p_R(R) = \alpha^{-1}\, 2R \exp\{-\frac{1}{\alpha}(R^2+r^2)\}\, (2\pi)^{-1} \int_0^{2\pi} \exp\{\frac{2}{\alpha} Rr \cos(\theta-\varphi)\}\, d\theta \quad ,$$

$$= \alpha^{-1} 2R \exp\{-\frac{1}{\alpha}(R^2+r^2)\} (2\pi)^{-1} \int_{-\varphi}^{2\pi-\varphi} \exp\{\frac{2}{\alpha} Rr \cos\theta\} d\theta$$

$$= \alpha^{-1} 2R \exp\{-\frac{1}{\alpha}(R^2+r^2)\} I_0\left(\frac{2Rr}{\alpha}\right) \qquad 0 < R < \infty \quad .$$

This is the Rice-Nakagami Density.

The marginal density of θ is not uniform and is given by:

$$p_\theta(\theta) = \int_0^\infty p_{R\theta}(R,\theta)\, dR$$

$$= (2\pi)^{-1} \exp(-r^2/\alpha) \left[1 + G\pi^{\frac{1}{2}} \exp(G^2) \{1 + \text{erf } G\}\right], \qquad 0 \leq \theta < 2\pi$$

where

$$G = \alpha^{-\frac{1}{2}} r \cos(\theta - \varphi) \quad .$$

The complement of the distribution function of R is given by:

$$\text{Prob } (R > \rho) = \int_\rho^\infty p_R(u)\, du$$

$$= 2\alpha^{-1} \int_\rho^\infty u \exp\{-\frac{1}{\alpha}(u^2+r^2)\} I_0\left(\frac{2ur}{\alpha}\right) du \quad .$$

Tables of this integral may be found in reference 3.

The solutions to this problem illustrate the asymmetry introduced by a noncentral initial position. As would be expected, they approach the corresponding solutions of Problems 1 or 1a as:

$$\frac{r}{R_{RMS}} = \frac{r}{\alpha^{\frac{1}{2}}} \Rightarrow 0 \quad .$$

PROBLEM 4

In this problem the initial position is uncertain but the density $p_{xy}(x, y)$ is specified. Otherwise the motion is as described in Problems 1 or 2.

From Problem 3,

$$p_{uv}(u,v\,|\,xy) = (2\pi)^{-1}\,\sigma^{-2}\exp\left[-\frac{1}{2\sigma^2}\,\{u-x)^2 + (v-y)^2\}\right] \quad .$$

Therefore,

$$p_{uv}(u,v) = \iint p_{uv}(u,v\,|\,x,y)\,p_{xy}(x,y)\,dx\,dy$$

$$= (2\pi)^{-1}\sigma^{-2}\iint \exp\left[-\frac{1}{2\sigma^2}\,\{(u-x)^2 + (v-y)^2\}\right] p_{xy}(x,y)\,dx\,dy \quad .$$

If (r,φ) are the polar coordinates of the initial position (x,y) ,

$$p_{r\varphi}(r,\varphi) = rp_{xy}(r\cos\varphi,\; r\sin\varphi)$$

$$p_r(r) \qquad = \int_0^{2\pi} p_{r\varphi}(r,\varphi)\,d\varphi \quad .$$

These densities may therefore be determined from the given density $p_{xy}(x,y)$.

Also from Problem 3,

$$p_R(R\,|\,r) = 2\alpha^{-1}\,R\exp\{-\frac{1}{\alpha}\,(R^2+r^2)\}\,I_0\left(\frac{2Rr}{\alpha}\right) \quad .$$

Therefore,

$$p_R(R) = \int_0^\infty p_R(R\,|\,r)\,p_r(r)\,dr$$

$$= 2\alpha^{-1}\int_0^\infty R\exp\{-\frac{1}{\alpha}\,(R^2+r^2)\}\,I_0\left(\frac{2Rr}{\alpha}\right)p_r(r)\,dr$$

and

$$\mathrm{Prob}(R > \rho) = 2\alpha^{-1}\int_\rho^\infty\int_0^\infty u\exp\{-\frac{1}{\alpha}\,(u^2+r^2)\}\,I_0\left(\frac{2ur}{\alpha}\right)p_r(r)\,dr\,du \quad .$$

These integral expressions permit the final position densities $p_{uv}(u, v)$ and $p_R(R)$ and the above complement of the final position distribution $P_R(R)$ to be determined for any given initial position density $p_{xy}(x, y)$, and in the following problem we shall obtain explicit results for the case where the initial position density is bivariate normal. Before considering this important case, however, we give an asymptotic result which holds for any initial position density.

It is clear that for sufficiently large values of R , the influence of the motion is minimal and the $\text{Prob}(R>\rho)$ is largely determined by the initial density $p_r(r)$. In contrast, for small values of R , the $\text{Prob}(R>\rho)$ will be low unless r is also small. This is evident from the exponential term in the integrand of the above expression for this probability. But if r is small, the asymmetry is small and $P_R(R)$ may be approximated by the Rayleigh Distribution.

A quantitative treatment of these issues yields the following:

$$
\begin{aligned}
p_R(R) &= p_r(R) && \text{for } R \gg \alpha^{\frac{1}{2}} \\
&= 2\alpha^{-1} R \exp(-R^2/\alpha) && \text{for } R \ll \alpha^{\frac{1}{2}} \quad .
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\text{Prob}\,(R > \rho) &= \int_\rho^\infty p_r(u)\,du \\
&= 1 - P_r(\rho) && \text{for } R \gg \alpha^{\frac{1}{2}} \\
&= \exp(-\rho^2/\alpha) && \text{for } R \ll \alpha^{\frac{1}{2}} \quad .
\end{aligned}
$$

PROBLEM 5

Problem 5 is the special case of Problem 4 where the initial position density $p_{xy}(x, y)$ is normal with parameters $\mu_x, \mu_y, \sigma_x^2, \sigma_y^2$ and ρ_{xy} .

Clearly the final position density $p_{uv}(u, v)$ may be obtained from the relationship given by the previous problem by performing the necessary integration. It is obvious, however, that, for this case, this density is also bivariate normal, and that its parameters $\mu_u, \mu_v, \sigma_u^2, \sigma_v^2$ and ρ_{uv} may be ascertained directly from the relationships:

$$u = x + u' \qquad u' = \sum_j x_j$$

$$v = y + v' \qquad v' = \sum_j y_j \quad .$$

We know from Problem 1 that u' and v' are normal variables with zero means and equal variances σ^2, where σ^2 may be obtained from the parameters which characterize the motion.

Therefore,

$$\mu_u = E(u) = E(x) = \mu_x$$

$$\mu_v = E(v) = E(y) = \mu_y \quad .$$

Also, since the motion is assumed independent of the initial position,

$$\sigma_u^2 = \sigma_x^2 + \sigma_{u'}^2 = \sigma_x^2 + \sigma^2$$

$$\sigma_v^2 = \sigma_y^2 + \sigma_{v'}^2 = \sigma_y^2 + \sigma^2$$

$$\rho_{uv} = \sigma_u^{-1} \sigma_v^{-1} \operatorname{cov}(uv)$$

$$= \sigma_u^{-1} \sigma_v^{-1} [E(uv) - E(u)E(v)]$$

$$= \sigma_u^{-1} \sigma_v^{-1} [E(xy) - E(x)E(y)]$$

$$= \sigma_u^{-1} \sigma_v^{-1} \operatorname{cov}(xy)$$

$$= \sigma_u^{-1} \sigma_v^{-1} \sigma_x \sigma_y \rho_{xy}$$

$$= \frac{\sigma_x \sigma_y \rho_{xy}}{(\sigma_x^2 + \sigma^2)^{\frac{1}{2}} (\sigma_y^2 + \sigma^2)^{\frac{1}{2}}} \quad .$$

This completes the specification of the final position density $p_{uv}(u, v)$, and from this, we have:

$$p_{R\theta}(R, \theta) = R\, p_{uv}(R \cos\theta,\ R \sin\theta) \quad .$$

To obtain the marginal density $p_R(R)$ it is convenient to rotate the coordinate system through the angle β where

$$\tan 2\beta = \frac{2\rho_{uv}\sigma_u\sigma_v}{\sigma_u^2-\sigma_v^2} \quad .$$

The new coordinates are then given in terms of the old by the relations:

$$u' = u \cos \beta + v \sin \beta$$

$$v' = -u \sin \beta + v \cos \beta$$

and the old in terms of the new by:

$$u = u' \cos \beta - v' \sin \beta$$

$$v = u' \sin \beta + v' \cos \beta \quad .$$

Also, we define $\mu_{u'}$ and $\mu_{v'}$ by the relations:

$$\mu_{u'} = \mu_u \cos \beta - \mu_v \sin \beta$$

$$\mu_{v'} = \mu_u \sin \beta + \mu_v \cos \beta$$

from which,

$$\mu_u = \mu_{u'} \cos \beta + \mu_{v'} \sin \beta$$

$$\mu_v = -\mu_{u'} \sin \beta + \mu_{v'} \cos \beta \quad .$$

Then

$$p_{u'v'}(u', v') = (2\pi)^{-1}(S_{u'}S_{v'})^{-\frac{1}{2}} \exp\left[-\frac{1}{2}\{(u'-\mu_{u'})^2/S_{u'} + (v'-\mu_{v'})^2/S_{v'}\}\right]$$

where

$$\frac{1}{S_{u'}}, \frac{1}{S_{v'}} = \frac{1}{2(1-\rho_{uv}^2)}\left[\frac{1}{\sigma_u^2} + \frac{1}{\sigma_v^2} \mp \left\{\left(\frac{1}{\sigma_u^2} + \frac{1}{\sigma_v^2}\right)^2 - \frac{4(1-\rho_{uv}^2)}{\sigma_u^2\sigma_v^2}\right\}^{\frac{1}{2}}\right] .$$

$$p_{R\theta}(R, \theta) = R\, p_{u'v'}\{R \cos(\theta-\beta),\ R \sin(\theta-\beta)\} \quad .$$

Thus,

$$p_R(R) = \int_0^{2\pi} R\, p_{u'v'}\{R\cos(\theta-\beta),\ R\sin(\theta-\beta)\}\, d\theta$$

$$= \int_{-\beta}^{2\pi-\beta} R\, p_{u'v'}\{R\cos\theta,\ R\sin\theta\}\, d\theta$$

$$= (2\pi)^{-1}(S_{u'}S_{v'})^{-\frac{1}{2}} R \int_{-\beta}^{2\pi-\beta} \exp\left[-\frac{1}{2}\{(R\cos\theta-\mu_{u'})^2/S_{u'} + (R\sin\theta-\mu_{v'})^2/S_{v'}\}\right] d\theta.$$

The integral on the right of this expression may be reduced to an infinite series of Modified Bessel Functions of the First Kind. Details of the reduction are similar to those used in the appendix of reference 1. When this reduction is carried out, we obtain

$$p_R(R) = (S_{u'}S_{v'})^{-\frac{1}{2}} R\exp(-q) \sum_{k=0}^{\infty} (-1)^k \epsilon_k I_k(f)\, I_{2k}(w) \cos 2k\gamma\ , \qquad 0 < R < \infty$$

where

$$\epsilon_k = 1 \quad \text{for } k = 0$$

$$= 2 \quad \text{for } k > 0$$

$$q(R) = \frac{\mu_{u'}^2}{2S_{u'}} + \frac{\mu_{v'}^2}{2S_{v'}} + \frac{R^2}{4}\left(\frac{1}{S_{u'}} + \frac{1}{S_{v'}}\right)$$

$$f(R) = \frac{R^2}{4}\left(\frac{1}{S_{u'}} - \frac{1}{S_{v'}}\right)$$

$$w(R) = R\left(\frac{\mu_{u'}^2}{S_{u'}^2} + \frac{\mu_{v'}^2}{S_{v'}^2}\right)^{\frac{1}{2}}$$

$$\gamma = \arctan \frac{\mu_{v'}/S_{v'}}{\mu_{u'}/S_{u'}}\ .$$

This is the Nakagami Density (reference 4).

Finally,

$$P_\theta(\theta) = \int_0^\infty p_{R\theta}(R, \theta)\, dR$$

$$= \int_0^\infty R\, p_{uv}(R \cos \theta,\ R \sin \theta)\, dR \quad .$$

A closed form expression for this density has been obtained only in special cases.

PROBLEM 5a

The statement of Problem 5a is identical to that of Problem 5. The difference lies in the treatment.

It is clear that part of the complexity of the results obtained in the preceding problem arises from the fact that a prespecified coordinate system was assumed. This, of course, is necessary if any information relating to the final position of the target is to be conveyed to someone else. It is relatively simple, however, to devise programs which will translate and rotate the coordinate system originally specified to one better adapted to the problem at hand and, on completion, reestablish the original reference. If a program of this type is available, it is advantageous to center the new coordinates at the mean of the initial position density (μ_x, μ_y) and to rotate them through the angle β where

$$\tan \beta = \frac{2\sigma_x \sigma_y \rho_{xy}}{\sigma_x^2 - \sigma_y^2} \quad .$$

Then, if (x', y') denotes the initial position with respect to the new coordinate system,

$$p_{x'y'}(x', y') = (2\pi)^{-1}(S_{x'} S_{y'})^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(x'^2/S_{x'} + y'^2/S_{y'})\right\}$$

where $S_{x'}, S_{y'}$ are given by:

$$\frac{1}{S_{x'}}, \frac{1}{S_{y'}} = \frac{1}{2(1-\rho_{xy}^2)}\left[\frac{1}{\sigma_x^2} + \frac{1}{\sigma_y^2} \mp \left\{\left(\frac{1}{\sigma_x^2} + \frac{1}{\sigma_y^2}\right)^2 - \frac{4(1-\rho_{xy}^2)}{\sigma_x^2 \sigma_y^2}\right\}^{\frac{1}{2}}\right] \quad .$$

Thus, with respect to the modified reference system, the density of the initial position is normal, but requires only two parameters, $S_{x'}$ and $S_{y'}$ for its complete specification.

Continuing as in Problem 5, but with primed variables to denote reference to the modified coordinate system, we obtain:

$$\mu_{u'} = \mu_{v'} = 0$$

$$\sigma_{u'}^2 = S_{x'} + \sigma^2$$

$$\sigma_{v'}^2 = S_{y'} + \sigma^2$$

$$\rho_{u'v'} = 0 \quad .$$

Thus,

$$p_{u'v'}(u', v') = p_{u'}(u')p_{v'}(v')$$

$$= (2\pi\sigma_{u'}\sigma_{v'})^{-1} \exp\{-u'^2/2\sigma_{u'}^2 - v'^2/2\sigma_{v'}^2\}.$$

$$p_{R'\theta'}(R', \theta') = R'p_{u'v'}(R'\cos\theta', R'\sin\theta')$$

$$= (2\pi\sigma_{u'}\sigma_{v'})^{-1} R' \exp\left\{-\frac{R'^2\cos^2\theta'}{2\sigma_{u'}^2} - \frac{R'^2\sin^2\theta'}{2\sigma_{v'}^2}\right\}$$

$$p_{R'}(R') = \int_0^{2\pi} p_{R'\theta'}(R', \theta')\, d\theta$$

$$= (2\pi\sigma_{u'}\sigma_{v'})^{-1} R' \int_0^{2\pi} \exp\left[-\frac{R'^2}{4\sigma_{u'}^2}(1+\cos 2\theta) - \frac{R'^2}{4\sigma_{v'}^2}(1-\cos 2\theta)\right] d\theta$$

$$= (\sigma_{u'}\sigma_{v'})^{-1} R' \exp\left[-\frac{R'^2}{4}\left(\frac{1}{\sigma_{u'}^2} + \frac{1}{\sigma_{v'}^2}\right)\right]$$

$$(2\pi)^{-1} \int_0^{2\pi} \exp\left[-R'^2\cos 2\theta\left(\frac{1}{4\sigma_{u'}^2} - \frac{1}{4\sigma_{v'}^2}\right)\right] d\theta$$

$$= (\sigma_{u'}\sigma_{v'})^{-1} R' \exp\left[-\frac{R'^2}{4}\left(\frac{1}{\sigma_{u'}^2} + \frac{1}{\sigma_{v'}^2}\right)\right] I_0\left\{\frac{R'^2}{4}\left(\frac{1}{\sigma_{u'}^2} - \frac{1}{\sigma_{v'}^2}\right)\right\}$$

$$0 < R' < \infty \quad .$$

Curves for this expression and its integral are given in reference 5.

$$p_{\theta'}(\theta') = \int_0^\infty p_{R'\theta'}(R', \theta')\, dR'$$

$$= (2\pi\sigma_{u'}\sigma_{v'})^{-1} \int_0^\infty R' \exp\left\{- \frac{R'^2\cos^2\theta'}{2\sigma_{u'}^2} - \frac{R'^2\sin^2\theta'}{2\sigma_{v'}^2}\right\} dR'$$

$$= (2\pi\sigma_{u'}\sigma_{v'})^{-1} \left\{\frac{\cos^2\theta'}{2\sigma_{u'}^2} + \frac{\sin^2\theta'}{2\sigma_{v'}^2}\right\}^{-1} \int_0^\infty u \exp(-u^2)\, du$$

$$= \frac{1}{2\pi} \frac{K}{K^2\cos^2\theta' + \sin^2\theta'} \qquad 0 \leq \theta' < 2\pi$$

where

$$K = \sigma_{u'}/\sigma_{v'} \quad .$$

Returning to the joint density of the final position, it is readily shown that curves of constant probability are ellipses centered at the new origin and with axes parallel to the modified coordinate axes. Their equations are given by:

$$\frac{u'^2}{\sigma_{u'}^2} + \frac{v'^2}{\sigma_{v'}^2} = C$$

for values of $C > 0$.

The parameter C may be related to a measure of confidence that the target lies within the ellipse which it defines. If this measure is denoted by P , then

$$C = -2 \log(1-P) \quad .$$

This relationship thus defines the ellipse corresponding to a prespecified value of P . Its area is given by:

$$\pi\sigma_{u'}\sigma_{v'}C = -2\pi\sigma_{u'}\sigma_{v'} \log(1-P)$$

and the lengths of the semi-axes by:

$$\sigma_{u'}\sqrt{C} \text{ and } \sigma_{v'}\sqrt{C} \quad .$$

These quantities are not changed when the original coordinate system is restored.

PROBLEM 6

In this problem the initial position is at the origin of the coordinate system but changes in course are nonuniform. The components of the motion are sufficiently large in number and homogeneous in composition to justify the application of the Central Limit Theorem. As before, each component of the motion is assumed to be independent, but, at least initially, we shall not require V_j and φ_j to be independent for a common value of j.

As before,

$$u = \int_0^T V(t) \cos \varphi(t) \; dt \quad ,$$

but

$$E(u) \neq \int_0^T E\{V(t)\} \, E\{\cos \varphi(t)\} \, dt$$

since $V(t)$ and $\varphi(t)$ are not independent. Even if these variables were independent for all t, since φ_j, and hence $\varphi(t)$, are nonuniform, in general

$$E\{\cos \varphi(t)\} \neq 0 \quad .$$

Similar arguments prevail for $E(v)$.

Thus, where the changes in course are nonuniform but the conditions of the Central Limit Theorem are met, the final position density $p_{uv}(u,v)$ is normal but the mean is not, in general, at the origin, having instead some value (μ_u, μ_v) depending on the joint densities $p_{V_j \varphi_j}(V_j, \varphi_j)$.

The variances σ_u^2 and σ_v^2 are obtained by taking appropriate expectations but are no longer expressible in terms of a parameter of the speed distribution alone. Finally, for the conditions of this problem, the covariance of u and v does not automatically vanish.

The joint density of the final position $p_{uv}(u,v)$ is thus bivariate normal of the most general type requiring the five parameters $\mu_u, \mu_v, \sigma_u^2, \sigma_v^2$, and ρ_{uv} for its complete specification. This being the case, however, all of the results of Problem 5 are also applicable to this problem.

$$p_{R\theta}(R, \theta) = R\, p_{uv}(R \cos \theta,\ R \sin \theta)$$

is readily obtained once the parameters of $p_{uv}(u, v)$ have been determined.

$p_R(R)$ has a Nakagami Density with parameters derived from those of $p_{uv}(u, v)$:

$$p_\theta(\theta) = \int_0^\infty p_{R\theta}(R, \theta)\, dR$$

$$= \int_0^\infty R\, p_{uv}(R \cos \theta,\ R \sin \theta)\, dR \quad .$$

As previously stated, a closed-form expression for this density is not available in the most general case. There is, however, a case of some practical interest in which a simpler expression results for the density $p_R(R)$ and the integral on the right of the above expression can be evaluated.

If V_j and φ_j are independent for all j, and φ_j is symmetrically distributed about its mean value, clearly $\varphi(t)$ also has this property.

Therefore,

$$\mu_v = E(v) = E \int_0^T V(t) \sin \varphi(t)\, dt$$

$$= \int_0^T E\{V(t)\}\, E\{\sin \varphi(t)\}\, dt$$

$$= 0$$

since

$$E\{\sin \varphi(t)\} = \int_0^{2\pi} \sin \varphi\, p_\varphi(\varphi)\, d\varphi$$

$$= \int_0^{\mu_\varphi} \sin \varphi\, p_\varphi(\varphi)\, d\varphi + \int_{\mu_\varphi}^{2\pi} \sin \varphi\, p_\varphi(\varphi)\, d\varphi$$

$$= \int_0^{\mu_\varphi} \sin \varphi\, p_\varphi(\varphi)\, d\varphi + \int_{\mu_\varphi}^{o} \sin\varphi\, p_\varphi(\varphi)\, d\varphi$$

$$= 0 \quad .$$

Therefore, setting $\mu_v = 0$ in the Nakagami Density, we obtain:

$$p_R(R) = (S_u S_v)^{-\frac{1}{2}} R \exp(-q) \sum_{k=1}^{\infty} (-1)^k \epsilon_k I_k(f) I_{2k}(w) \quad , \qquad 0 < R < \infty$$

where

$$\epsilon_k = 1 \qquad \text{for } k = 0$$
$$= 2 \qquad \text{for } k > 0$$

$$q((R) = \frac{\mu_u^{\,2}}{2S_u} + \frac{R^2}{4}\left(\frac{1}{S_u} + \frac{1}{S_v}\right)$$

$$f(R) = \frac{R^2}{4}\left(\frac{1}{S_u} - \frac{1}{S_v}\right)$$

$$w(R) = R\,\frac{\mu_u}{S_u} \quad .$$

$$p_\theta(\theta) = \frac{K \exp\{-\frac{1}{2}B^2(1+K^2)\}}{2\pi(K^2\cos^2\theta+\sin^2\theta)} \{1 + G\pi^{\frac{1}{2}} \exp(G^2)(1 + \operatorname{erf} G)\} \; , \quad 0 \le \theta < 2\pi$$

where

$$G(\theta) = BK \cos\theta \left[\frac{1 + K^2}{2(K^2\cos^2\theta+\sin^2\theta)}\right]^{\frac{1}{2}}$$

$$B = \frac{\mu_u}{(\sigma_u^{\,2} + \sigma_v^{\,2})^{\frac{1}{2}}}$$

$$K = \sigma_v/\sigma_u \quad .$$

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